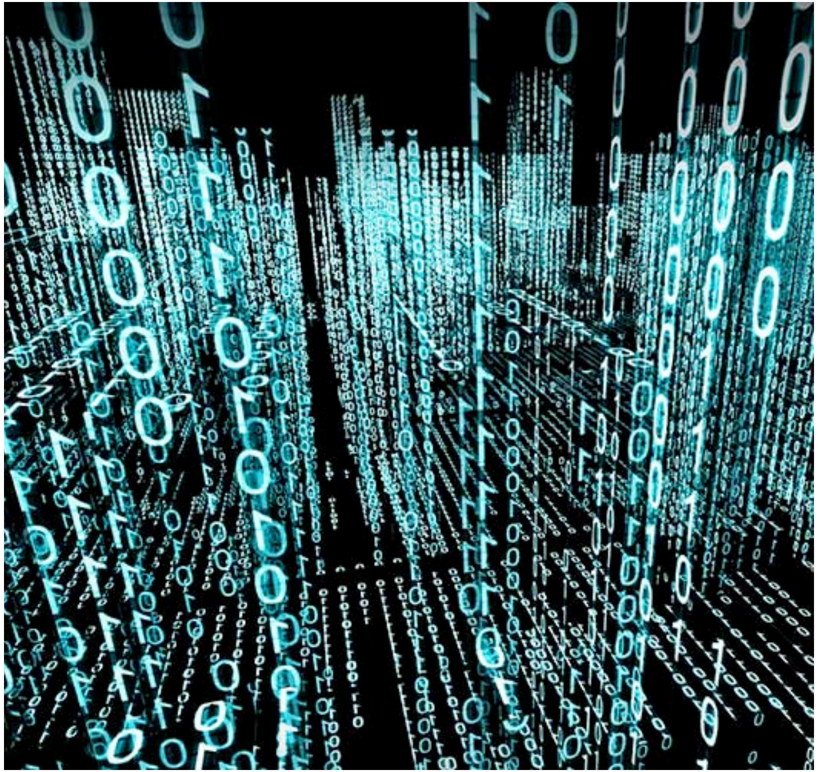


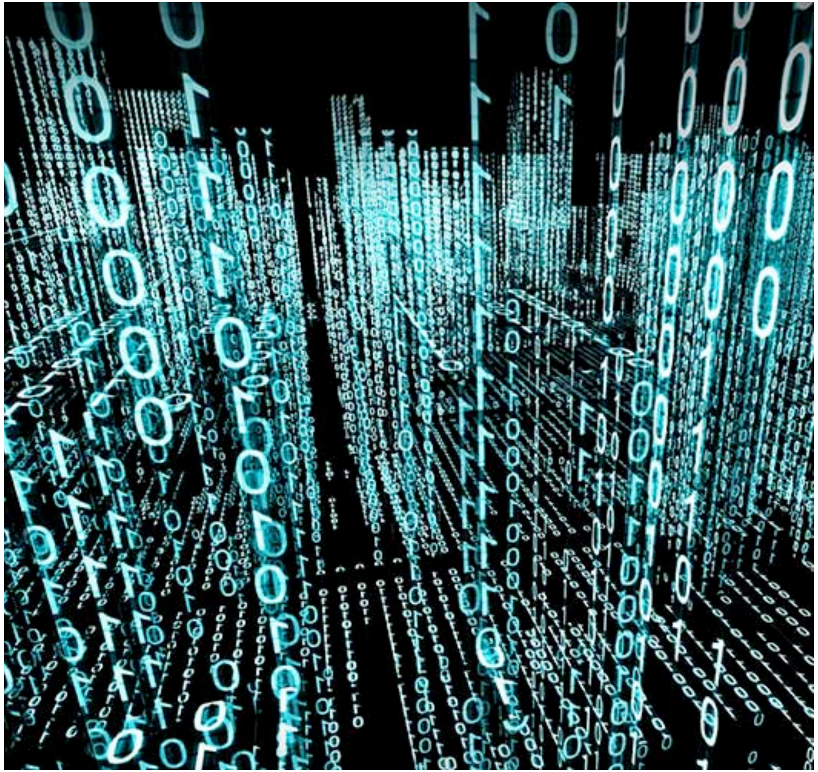
EECS 151/251A
Spring 2024
Digital Design and Integrated Circuits

Instructor:
John Wawrzynek

Lecture 6: Combinational Logic Representations

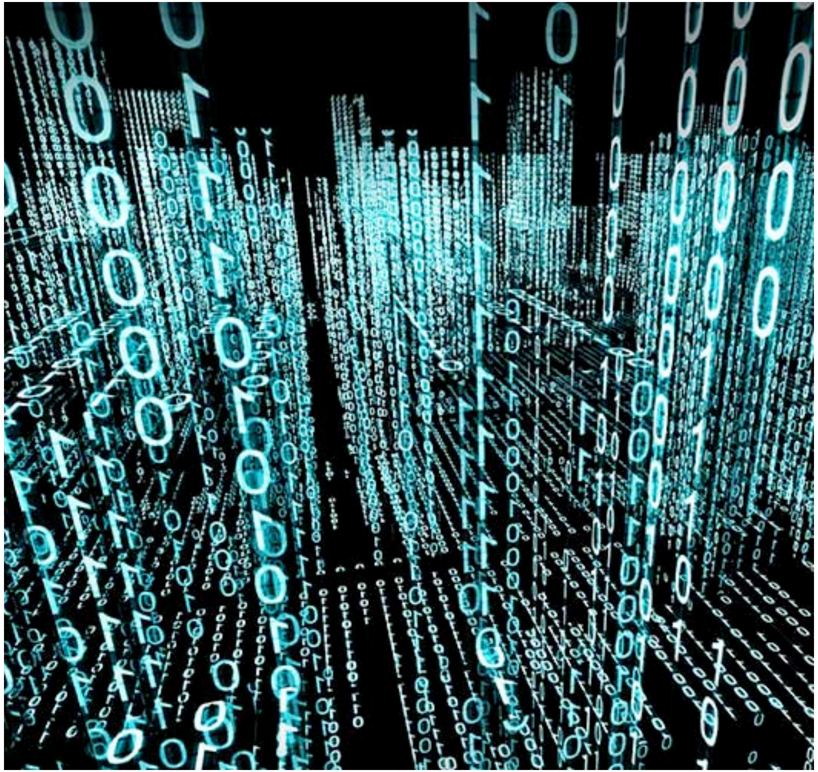
Announcements

- **PS 2 due tonight!**
- HW 3 posted, due in a week.



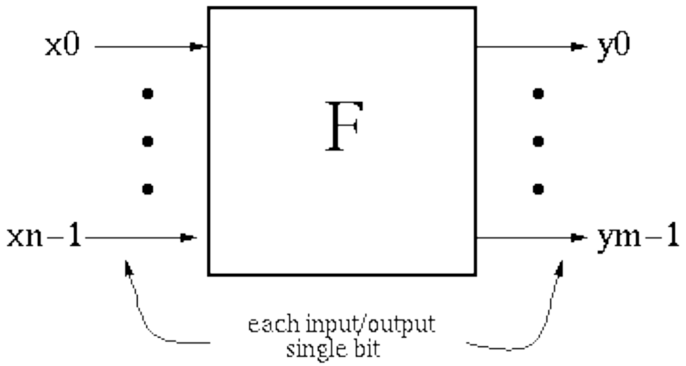
Outline

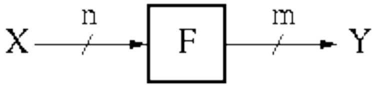
- Three representations for combinational logic:
  - truth tables,
  - graphical (logic gates), and
  - algebraic equations
- Boolean Algebra
- Boolean Simplification
- Multi-level Logic,
- NAND/NOR
- XOR



Representations of Combinational Logic

Combinational Logic (CL) Defined



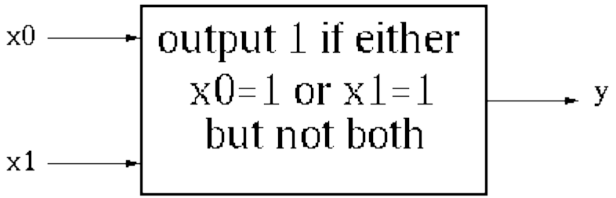


$y_i = f_i(x0 , ... , xn\text{-}1)$, where x, y take on values {0,1}.

Y is a function of only X, i.e., it is a “pure function”.

- If we change X then Y will change immediately (well almost!).
  - There is an ***implementation dependent*** delay from X to Y.
  - Y is a function of nothing other than the current inputs values.

CL Block Example #1



Truth Table Description:

x0	x1	y
0	0	0
0	1	1
1	0	1
1	1	0

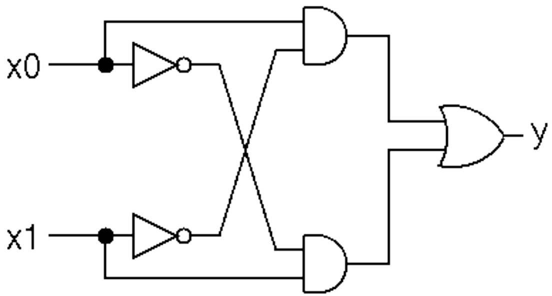
Boolean Equation:

y_0 = (x_0 AND not(x_1))

OR (not(x_0) AND x_1)

$y_0 = x_0x_1' + x_0'x_1$

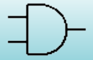
Gate Representation:



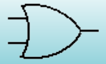
Boolean Algebra/Logic Circuits

- Why are they called “logic circuits”?
- Logic: The study of the principles of reasoning.
- The 19th Century Mathematician, George Boole, developed a math. system (algebra) involving logic, Boolean Algebra.
- His variables took on TRUE, FALSE
- Later Claude Shannon (father of information theory) showed (in his Master's thesis!) how to map Boolean Algebra to digital circuits:
- Primitive functions of Boolean Algebra:

a b	AND
0 0	0
0 1	0
1 0	0
1 1	1



a b	OR
0 0	0
0 1	1
1 0	1
1 1	1



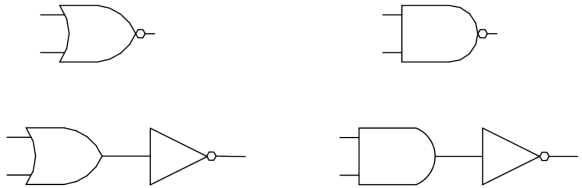
a	NOT
0	0
1	0

Why was this novel/innovative?

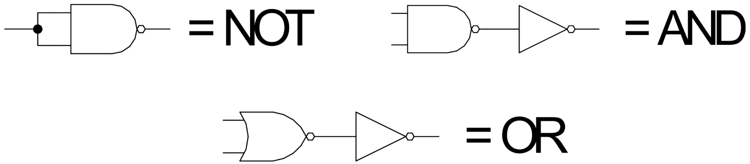
Other logic functions of 2 variables (x,y)

x y	f0	f1														
00	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
01	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
10	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
	0	AND		X		Y	⊕	OR	NOR	XNOR					NAND	1

Look at NOR and NAND:

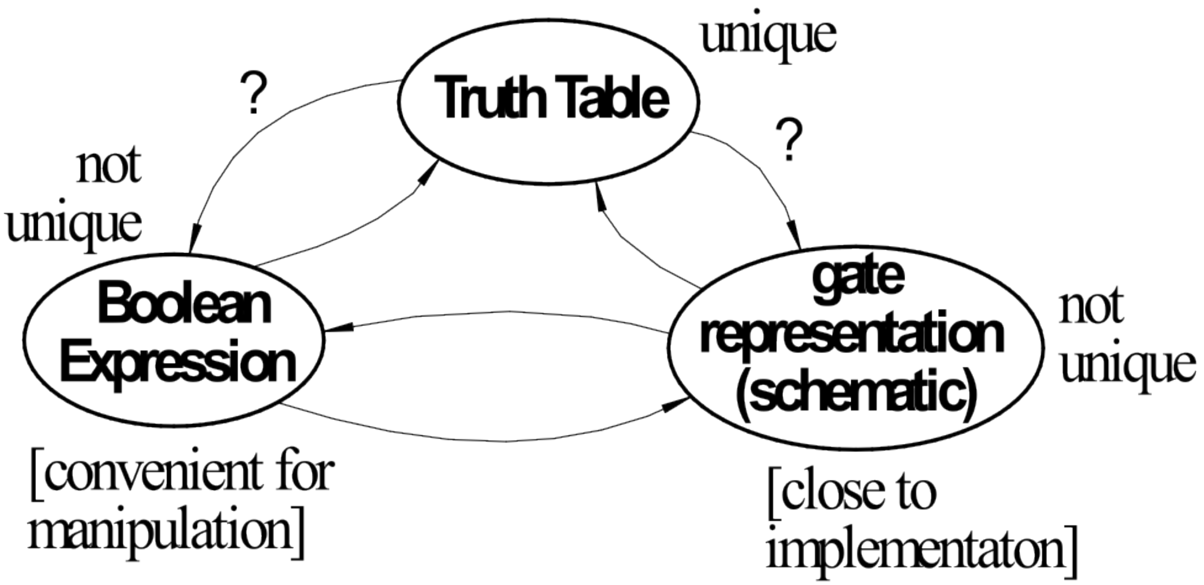


- Theorem: Any Boolean function that can be expressed as a truth table can be expressed using NAND and NOR.
  - Proof sketch:



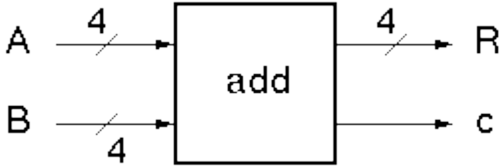
  - How would you show that either NAND or NOR is sufficient?

Relationship Among Representations



* Theorem: Any Boolean function that can be expressed as a truth table can be written as an expression in Boolean Algebra using AND, OR, NOT.

CL Block Example – 4 Bit Adder - where decomposition helps



R = A + B,
c is carry out

- Truth Table Representation:

a3	a2	a1	a0	b3	b2	b1	b0	r3	r2	r1	r0	c
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	0	1	1	0	0	1	1	0
0	0	0	0	0	1	0	0	0	1	0	0	0
⋮												
0	0	1	0	0	0	1	0	0	1	0	0	0
0	0	1	0	0	0	1	1	0	1	0	1	0
⋮												
0	0	0	1	1	1	1	1	0	0	0	0	1

256 rows!

In general: 2^n rows for n inputs.

Is there a more efficient (compact) way to specify this function?

4-bit Adder Example

- Motivate the adder circuit design by hand addition:

	a3	a2	a1	a0
+	b3	b2	b1	b0
c	r3	r2	r1	r0

- Add a0 and b0 as follows:

a	b	r	c
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

carry to next stage

$r = a \text{ XOR } b = a \oplus b$

$c = a \text{ AND } b = ab$

	a3	a2	a1	a0
+	b3	b2	b1	b0
c	r3	r2	r1	r0

- Add a1 and b1 as follows:

ci	a	b	r	co
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

$r = a \oplus b \oplus c_i$

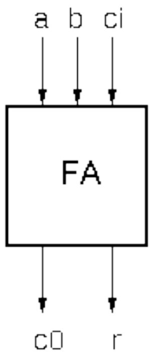
$co = ab + ac_i + bc_i$

4-bit Adder Example

- In general:

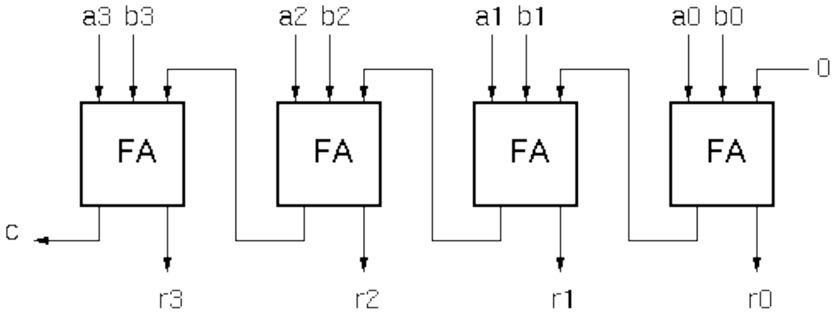
$$r_i = a_i \oplus b_i \oplus c_{in}$$

$$c_{out} = a_i c_{in} + a_i b_i + b_i c_{in} = c_{in}(a_i + b_i) + a_i b_i$$



“Full adder cell”

- Now, the 4-bit adder:



“ripple” adder

Full-adder (FA) cell example

- Graphical Representation of FA-cell

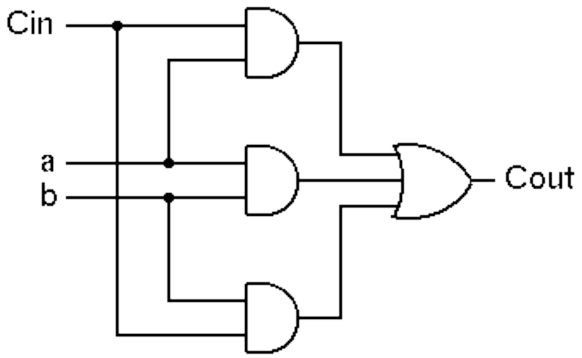
  $r_i = a_i \oplus b_i \oplus c_{in}$

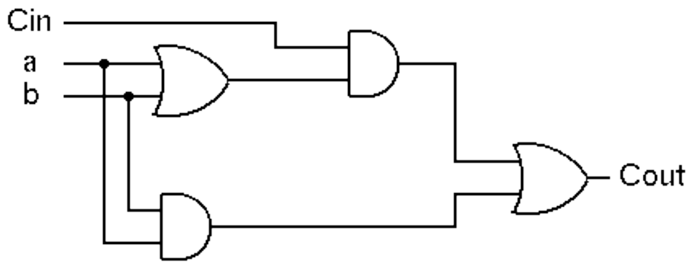
  $c_{out} = a_i c_{in} + a_i b_i + b_i c_{in}$

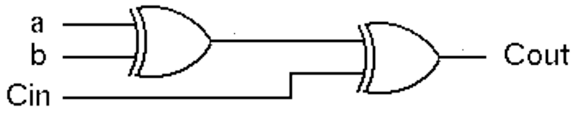
- Alternative Implementation (with only 2-input gates):

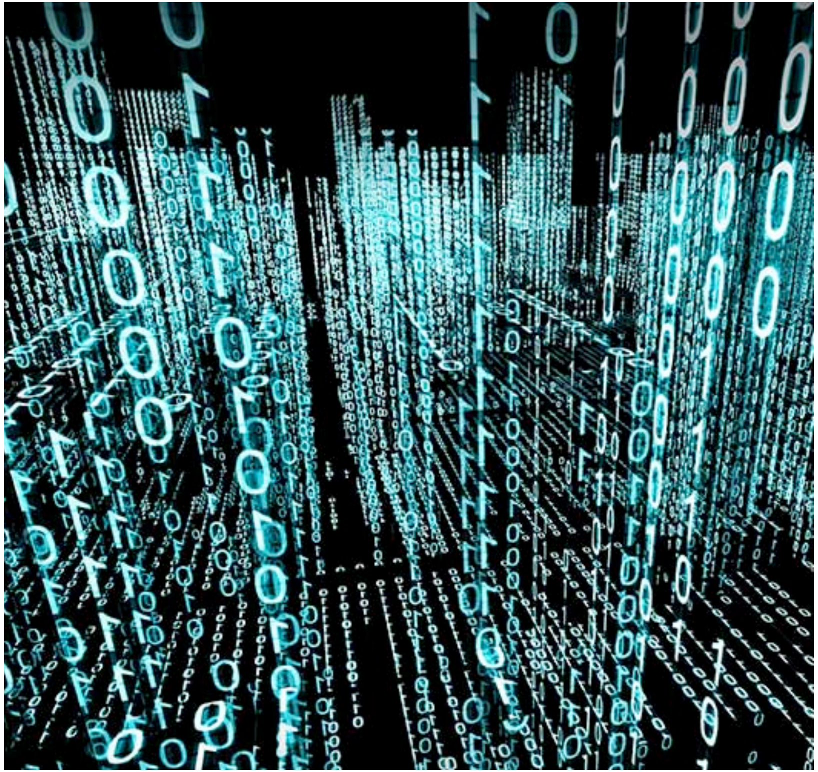
  $r_i = (a_i \oplus b_i) \oplus c_{in}$

  $c_{out} = c_{in}(a_i + b_i) + a_i b_i$









Boolean Algebra

Boolean Algebra

Set of elements B, binary operators $\{+, \bullet\}$, unary operation $\{'\}$, such that the following axioms hold:

1. B contains at least two elements a, b such that $a \neq b$.
2. Closure: a, b in B,

   $a + b$ in B, $a \bullet b$ in B, a' in B.
3. Communitive laws:

   $a + b = b + a, \quad a \bullet b = b \bullet a.$
4. Identities: 0, 1 in B

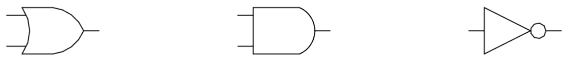
   $a + 0 = a, \quad a \bullet 1 = a.$
5. Distributive laws:

   $a + (b \bullet c) = (a + b) \bullet (a + c), \; a \bullet (b + c) = a \bullet b + a \bullet c.$
6. Complement:

   $a + a' = 1, \; a \bullet a' = 0.$

$B = \{0,1\}, + = \text{OR}, \bullet = \text{AND}, \; ' = \text{NOT}$

is a valid Boolean Algebra.



00	0
01	1
10	1
11	1

00	0
01	0
10	0
11	1

0	1
1	0

Some Laws (theorems) of Boolean Algebra

Duality: A dual of a Boolean expression is derived by interchanging OR and AND operations, and interchanging 0s and 1s (the variables are left unchanged).

$$\{F(x_1, x_2, \ldots, x_n, 0, 1, +, \bullet)\}^D = \{F(x_1, x_2, \ldots, x_n, 1, 0, \bullet, +)\}$$

Any law that is true for an expression is also true for its dual.

Operations with 0 and 1:

x + 0 = x   **x * 1 = x**

x + 1 = 1   **x * 0 = 0**

Idempotent Law:

x + x = x   **x x = x**

Involution Law:

(x')' = x

Laws of Complementarity:

x + x' = 1   **x x' = 0**

Commutative Law:

x + y = y + x   **x y = y x**

Some Laws (theorems) of Boolean Algebra (cont.)

Associative Laws:

$$(x + y) + z = x + (y + z) \qquad\qquad x\,y\,z = x\,(y\,z)$$

Distributive Laws:

$$x\,(y + z) = (x\,y) + (x\,z) \qquad\qquad x + (y\,z) = (x + y)(x + z)$$

"Simplification" Theorems:

$$\boxed{x\,y + x\,y' = x} \qquad\qquad (x + y)\,(x + y') = x$$

$$x + x\,y = x \qquad\qquad x\,(x + y) = x$$

$$x + x'y = x + y \qquad\qquad x(x' + y) = xy$$

DeMorgan's Law:

$$(x + y + z + \ldots)' = x'y'z' \qquad\qquad (x\,y\,z\,\ldots)' = x' + y' + z'$$

Theorem for Multiplying and Factoring:

$$(x + y)\,(x' + z) = x\,z + x'\,y$$

Consensus Theorem:

$$x\,y + y\,z + x'\,z = (x + y)\,(y + z)\,(x' + z)$$

$$x\,y + x'\,z = (x + y)\,(x' + z)$$

DeMorgan's Law

$$(x + y)' = x' y'$$

Exhaustive Proof

x	y	x'	y'	(x + y)'	x'y'
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

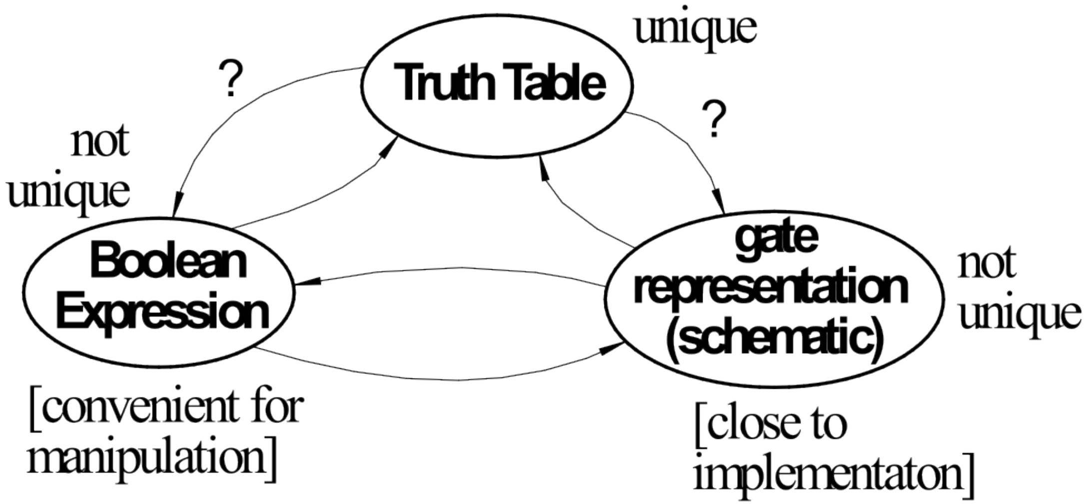
$$(x y)' = x' + y'$$

Exhaustive Proof

x	y	x'	y'	(x y)'	x' + y'
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

Relationship Among Representations

* Theorem: Any Boolean function that can be expressed as a truth table can be written as an expression in Boolean Algebra using AND, OR, NOT.



How do we convert from one to the other?

Canonical Forms

- Standard form for a Boolean expression - unique algebraic expression directly from a true table (TT) description.
- Two Types:
  - * **Sum of Products (SOP)**
  - * **Product of Sums (POS)**

- Sum of Products (disjunctive normal form, minterm expansion). Example:

Minterms	a	b	c	f	f′
a'b'c'	0	0	0	0	1
a'b'c	0	0	1	0	1
a′bc'	0	1	0	0	1
a′bc	0	1	1	1	0
ab′c'	1	0	0	1	0
ab′c	1	0	1	1	0
abc′	1	1	0	1	0
abc	1	1	1	1	0

One product (and) term for each 1 in f:

$f = a'bc + ab'c' + ab'c + abc' + abc$

$f' = a'b'c' + a'b'c + a'bc'$

(enumerate all the ways the function could evaluate to 1)

What is the cost?

Sum of Products (cont.)

Canonical Forms are usually not minimal:

Our Example:

```
f  = a'bc + ab'c' + ab'c + abc' +abc    [xy' + xy = x]
   = a'bc + ab' + ab                    [xy' + xy = x]
   = a'bc + a                           [x'y + x = y + x]
   = a + bc

f' = a'b'c' + a'b'c + a'bc'             [xy' + xy = x]
   = a'b' + a'bc'                       [x(y + z) = xy + xz]
   = a' ( b' + bc' )                    [x + x'y = x + y]
   = a' ( b' + c' )                     [x(y + z) = xy + xz]
   = a'b' + a'c'
```

Canonical Forms

- Product of Sums (conjunctive normal form, maxterm expansion). Example:

maxterms	a	b	c	f	f'
a+b+c	0	0	0	0	1
a+b+c′	0	0	1	0	1
a+b′+c	0	1	0	0	1
a+b′+c'	0	1	1	1	0
a′+b+c	1	0	0	1	0
a′+b+c′	1	0	1	1	0
a′+b'+c	1	1	0	1	0
a′+b′+c'	1	1	1	1	0

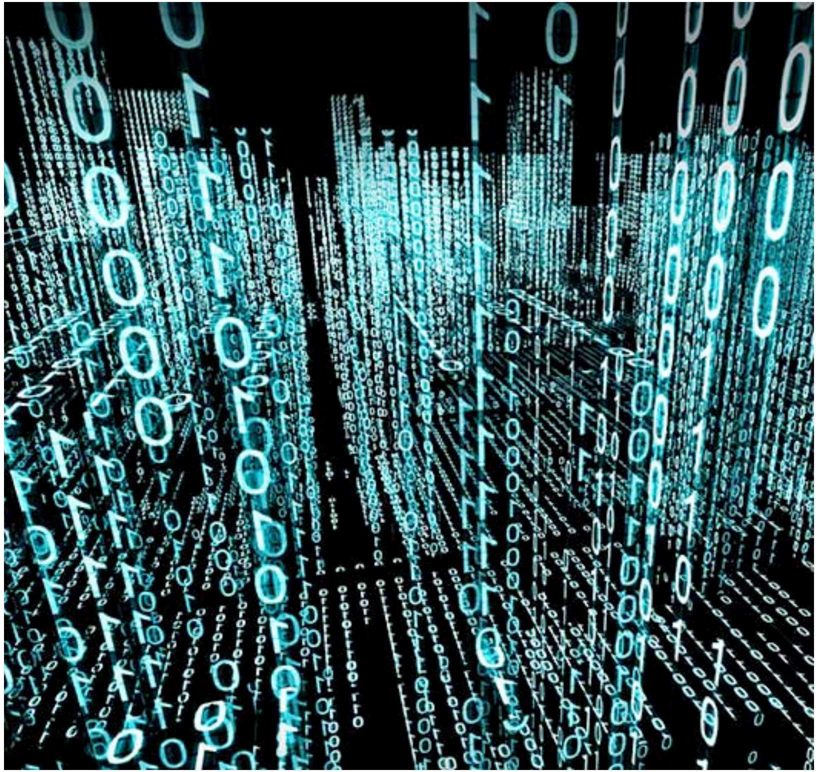
*One sum (**or**) term for each **0** in f:*

f = (a+b+c)(a+b+c')(a+b'+c)

f' = (a+b'+c')(a'+b+c)(a'+b+c')
(a'+b'+c)(a+b+c')

(enumerate all the ways the function could evaluate to 0)

What is the cost?



Boolean Simplification

Algebraic Simplification Example

Ex: full adder (FA) carry out function (in canonical form):

Cout = a'bc + ab'c + abc' + abc

ci	a	b	r	co
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Algebraic Simplification

$$
\begin{aligned}
Cout &= a'bc + ab'c + abc' + abc \\
&= a'bc + ab'c + abc' + abc + abc \\
&= a'bc + abc + ab'c + abc' + abc \\
&= (a' + a)bc + ab'c + abc' + abc \\
&= (1)bc + ab'c + abc' + abc \\
&= bc + ab'c + abc' + abc + abc \\
&= bc + ab'c + abc + abc' + abc \\
&= bc + a(b' + b)c + abc' + abc \\
&= bc + a(1)c + abc' + abc \\
&= bc + ac + ab(c' + c) \\
&= bc + ac + ab(1) \\
&= bc + ac + ab
\end{aligned}
$$

Outline for remaining CL Topics

- K-map method of two-level logic simplification
- Multi-level Logic
- NAND/NOR networks
- EXOR revisited

Algorithmic Two-level Logic Simplification

Key tool: <u>The Uniting Theorem:</u>

$$xy' + xy = x\,(y' + y) = x\,(1) = x$$

ab	f
00	0
01	0
10	1
11	1

$$f = ab' + ab = a(b'+b) = a$$

b values change within the on-set rows

a values don't change

b is eliminated, a remains

ab	g
00	1
01	0
10	1
11	0

$$g = a'b'+ab' = (a'+a)b' = b'$$

b values stay the same

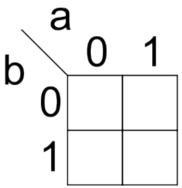
a values changes

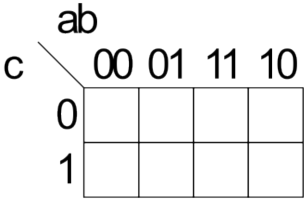
b' remains, a is eliminated

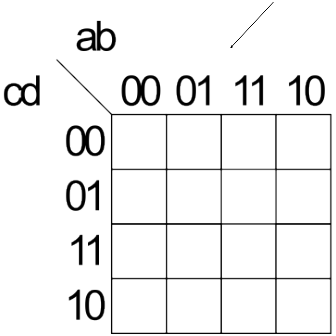
Karnaugh Map Method

- K-map is an alternative method of representing the TT and to help visual the adjacencies.

Note: "gray code" labeling.







5 & 6 variable k-maps possible

Karnaugh Map Method

- Adjacent groups of 1's represent product terms

b \ a	0	1
0	0	1
1	0	1

$f = a$

b \ a	0	1
0	1	1
1	0	0

$g = b'$

c \ ab	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$cout = ab + bc + ac$

c \ ab	00	01	11	10
0	0	0	1	1
1	0	0	1	1

$f = a$

K-map Simplification

1. Draw K-map of the appropriate number of variables (between 2 and 6)
2. Fill in map with function values from truth table.
3. Form groups of 1's.
   - ✓ Dimensions of groups must be even powers of two (1x1, 1x2, 1x4, …, 2x2, 2x4, …)
   - ✓ <u>Form as large as possible groups and as few groups as possible</u>.
   - ✓ Groups can overlap (this helps make larger groups)
   - ✓ Remember K-map is periodical in all dimensions (groups can cross over edges of map and continue on other side)
4. For each group write a product term.
   - the term includes the "constant" variables (use the uncomplemented variable for a constant 1 and complemented variable for constant 0)
5. Form Boolean expression as sum-of-products.

K-maps (cont.)

c \ ab	00	01	11	10
0	1	0	0	1
1	0	0	1	1

$f = b'c' + ac$

cd \ ab	00	01	11	10
00	1	0	0	1
01	0	1	0	0
11	1	1	1	1
10	1	1	1	1

$f = c + a'bd + b'd'$

(bigger groups are better)

Product-of-Sums K-map

1. Form groups of 0’s instead of 1’s.
2. For each group write a sum term.
   - the term includes the “constant” variables (use the uncomplemented variable for a constant 0 and complemented variable for constant 1)
3. Form Boolean expression as product-of-sums.

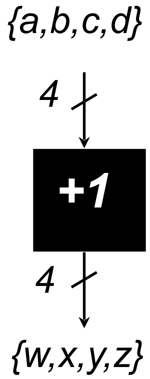
cd \ ab	00	01	11	10
00	1	0	0	1
01	0	1	0	0
11	1	1	1	1
10	1	1	1	1

$$f = (b' + c + d)(a' + c + d')(b + c + d')$$

BCD incrementer example

Binary Coded Decimal

	a	b	c	d	w	x	y	z
0	0	0	0	0	0	0	0	1
1	0	0	0	1	0	0	1	0
2	0	0	1	0	0	0	1	1
3	0	0	1	1	0	1	0	0
4	0	1	0	0	0	1	0	1
5	0	1	0	1	0	1	1	0
6	0	1	1	0	0	1	1	1
7	0	1	1	1	1	0	0	0
8	1	0	0	0	1	0	0	1
9	1	0	0	1	0	0	0	0
	1	0	1	0	-	-	-	-
	1	0	1	1	-	-	-	-
	1	1	0	0	-	-	-	-
	1	1	0	1	-	-	-	-
	1	1	1	0	-	-	-	-
	1	1	1	1	-	-	-	-



BCD Incrementer Example

- Note one map for each output variable.
- Function includes “don’t cares” (shown as “-” in the table).
  - These correspond to places in the function where we don’t care about its value, because we don’t expect some particular input patterns.
  - We are free to assign either 0 or 1 to each don’t care in the function, as a means to increase group sizes.
- In general, you might choose to write product-of-sums or sum-of-products according to which one leads to a simpler expression.

BCD incrementer example

w

cd \ ab	00	01	11	10
00	0	0	-	1
01	0	0	-	0
11	0	1	-	-
10	0	0	-	-

x

cd \ ab	00	01	11	10
00	0	1	-	0
01	0	1	-	0
11	1	0	-	-
10	0	1	-	-

y

cd \ ab	00	01	11	10
00	0	0	-	0
01	1	1	-	0
11	0	0	-	-
10	1	1	-	-

z

cd \ ab	00	01	11	10
00	1	1	-	1
01	0	0	-	0
11	0	0	-	-
10	1	1	-	-

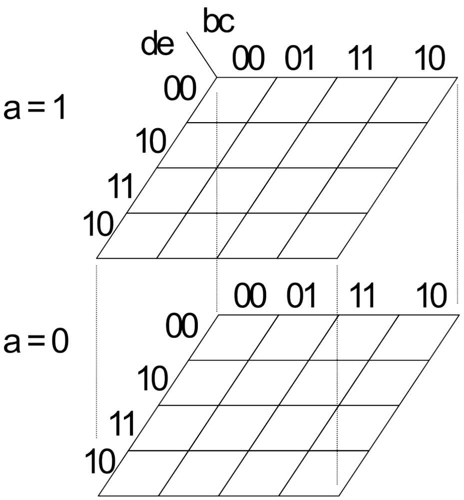
$w =$

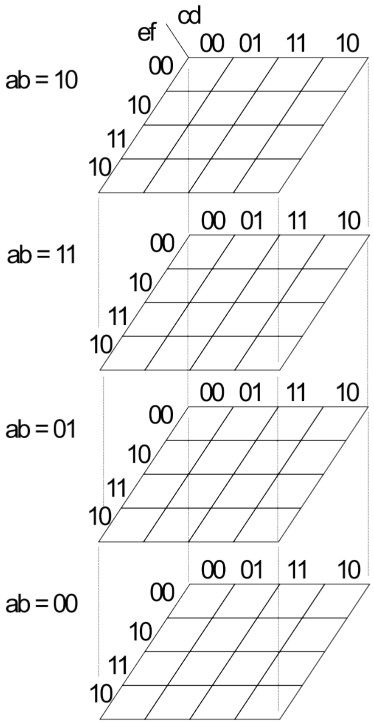
$x =$

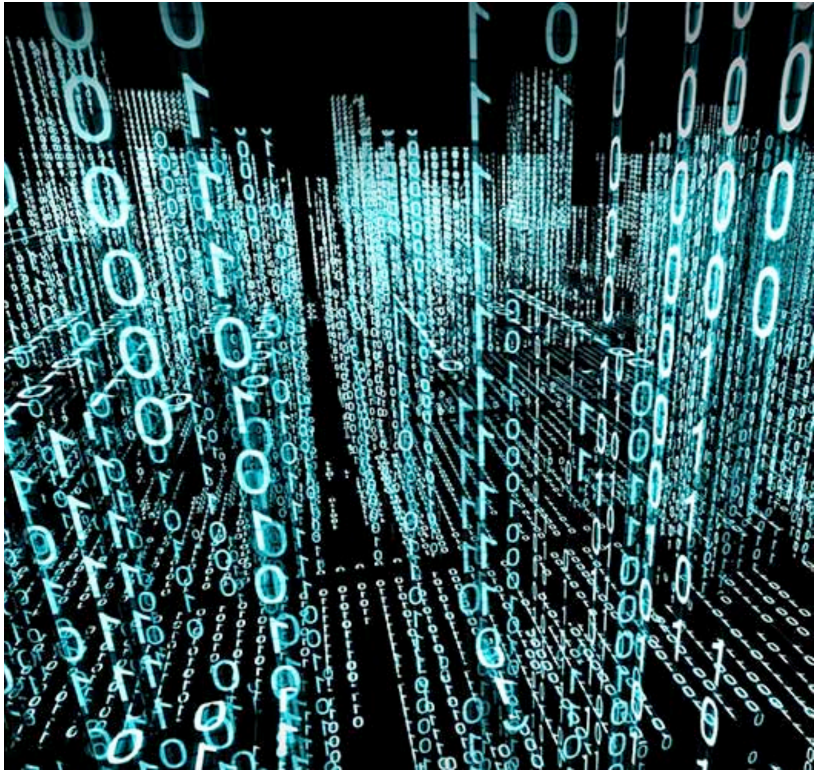
$y =$

$z =$

Higher Dimensional K-maps







Boolean Simplification – Multi-level Logic

Multi-level Combinational Logic

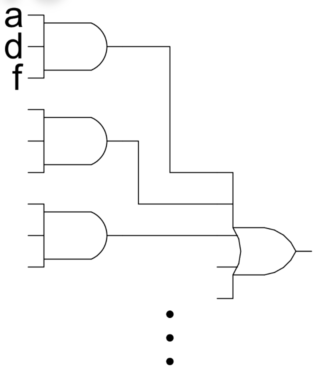
- Example: reduced sum-of-products form

  `x = adf + aef + bdf + bef + cdf + cef + g`
- Implementation in 2-levels with gates:

  cost: 1 7-input OR, 6 3-input AND

  => ~50 transistors

  delay: 3-input OR gate delay + 7-input AND gate delay



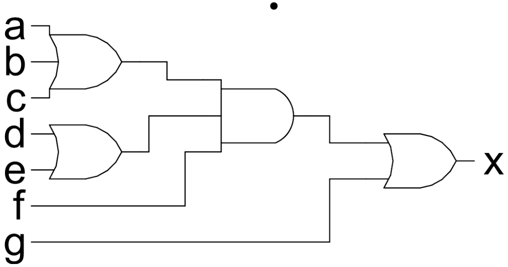
- Factored form:

  `x = (a + b +c)(d + e)f + g`

  cost: 1 3-input OR, 2 2-input OR, 1 3-input AND

  => ~20 transistors

  delay: 3-input OR + 3-input AND + 2-input OR



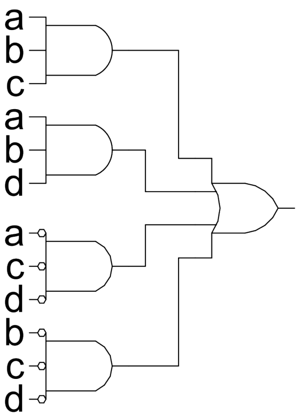
Footnote: NAND would be used in place of all ANDs and ORs.

Which is faster?

In general: Using multiple levels (more than 2) will reduce the cost. Sometimes also delay. Sometimes a tradeoff between cost and delay.

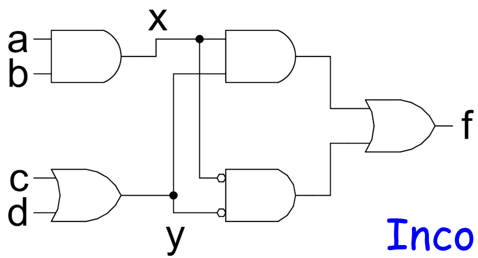
Multi-level Combinational Logic

Another Example: `F = abc + abd +a'c'd' + b'c'd'`



let `x = ab   y = c+d`

`f = xy + x'y'`



Incorporates fanout.

No convenient hand methods exist for multi-level logic simplification:

a) CAD Tools use sophisticated algorithms and heuristics

Guess what? These problems tend to be NP-complete

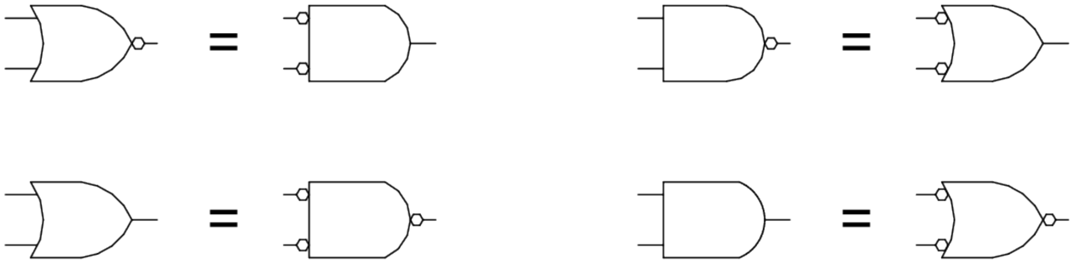
b) Humans and tools often exploit some special structure (example adder)

NAND-NAND & NOR-NOR Networks

DeMorgan's Law Review:

(a + b)' = a' b'   (a b)' = a' + b'

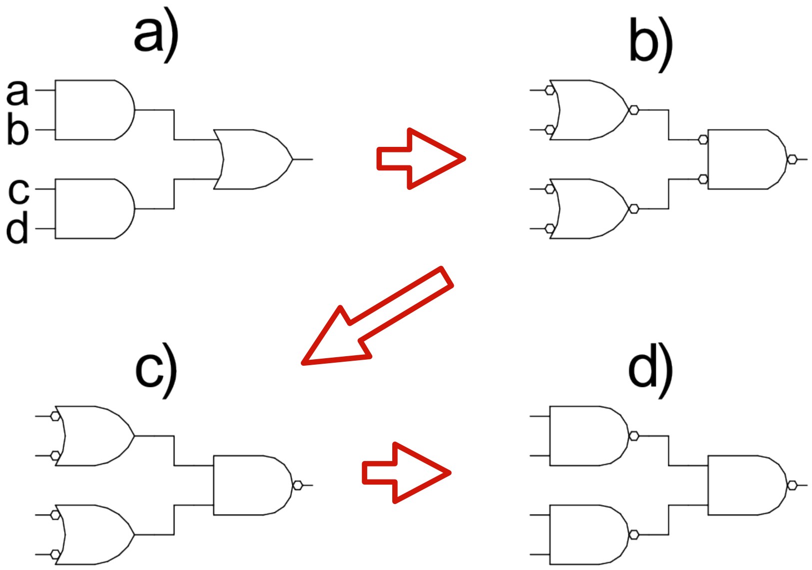
a + b = (a' b')'   (a b) = (a' + b')'



push bubbles or *introduce in pairs* or *remove pairs:*   *(x')' = x*

NAND-NAND & NOR-NOR Networks

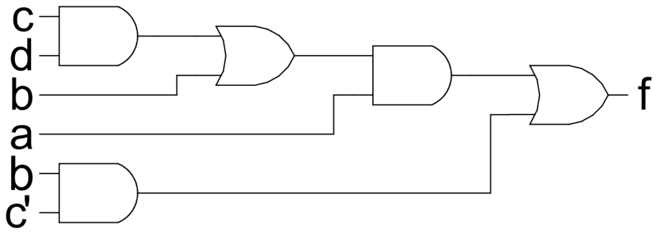
- Mapping from AND/OR to NAND/NAND



Multi-level Networks

Convert to NANDs:

$F = a(b + cd) + bc'$

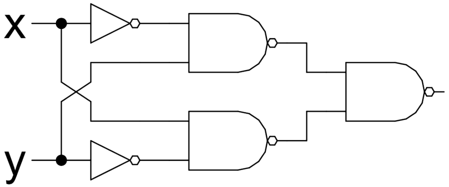


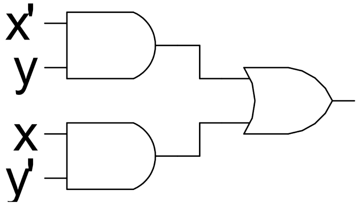
EXOR Function Implementations

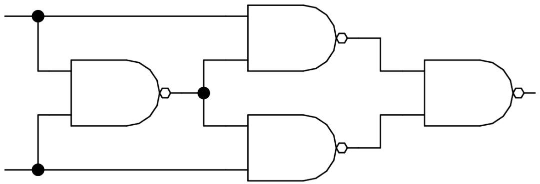
Parity, addition mod 2

$x \oplus y = x'y + xy'$

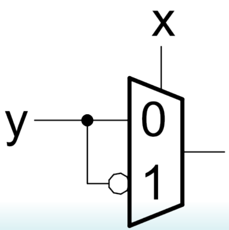
x	y	xor	xnor
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1







Another approach:



if x=0 then y else y'